2020 Online Physics Olympiad (OPhO): Invitational Contest Partial Solutions

OPhO Lead Organizers and Writers: Adithya Balachandran, Eddie Chen, Ashmit Dutta, Ethan Hu, Viraj Jayam, Rishab Parthasarathy, Kushal Thaman, Max Wang, QiLin Xue

OPhO Website Developer: Ben Oommen

Advisors: Dr. Branislav Kisačanin, Kye Shi, Kevin Zhou

Problem 1: A Party Trick (20 pts)

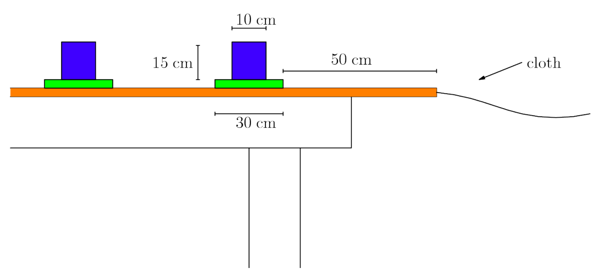
It is a well known party trick that by pulling the tablecloth very quickly and suddenly, the plates on top of the table can stay nearly in place.

(a) **(2 pt)** Start with a circular table $D = 1$ m in diameter with a very flat and small (and dimensionless for now) plate in the centre. How fast must you pull the tablecloth so the plate remains on the table? Assume the static friction coefficient is $\mu_s = 0.5$ and kinetic friction is $\mu_k = 0.3$ (for any contacts with the tablecloth) and that the tablecloth accelerates instantaneously. The tablecloth does not overhang the table.

(b) **(4 pts)** Mythbusters famously attempted to replicate the same trick with a giant tablecloth and a motorbike. We will simplify the experiment by assuming that the table was only set with plates placed $d = 0.5$ m apart from each other and from both ends. Assuming a $\ell = 7$ m long table, how fast must the bike be travelling to successfully carry out this experiment? The mass of the cloth and plates are negligible compared to the mass of the motorbike. Assume small plates and that the tablecloth does not overhang the table.

(c) **(6 pts)** A young boy places his toy car in the centre on the table from part (b). He believes that since it has wheels, it will stay on the table easier. Confirm or deny this statement and find the significant speeds that the tablecloth is pulled at that would cause the car to stay on the table. Assume the car (including wheels) has mass $2m$, and that each wheel is a uniform disk of mass $m/4$. The car is small in comparison to the table.

(d) **(8 pts)** We run the experiment one last time with the glass, with same mass as a plate, placed on top of each plate on the tablecloth, which in turn, is on a frictionless table. The static and kinetic coefficient between the glass and the plate are $\mu'_s = 0.30$ and $\mu'_k = 0.15$ respectively. How fast must the tablecloth be pulled so that the glasses stay completely on the plate and the plates stay completely on the table? Do not assume that the either plate or glass is dimensionless this time. A diagram is provided below:



Solution 1: TBD

Problem 2: Solar Sails (26 pts)

Much research is being done on the possibility of using solar sails to reach far away reaches in out galaxy. This is a method of propulsion that use light from the sun to exert a pressure on what is usually a large mirror. The sails themselves are often made of a thin reflective film. Compared to traditional spacecraft, while these sails have very limited payloads, they offer long operational lifetimes and are relatively low cost. The most significant advantage is speed: since solar sails do not depend on onboard propellant, they can travel much faster than a standard rocket, with the possibility of reaching a significant percentage of the speed of light. For all parts, assume the solar sails are only under the influence of gravity from the sun only. In addition, you may define necessary variables such as the mass and power output of the sun.

(a) **(2 pt)** Assume the solar sails are not revolving around the sun. Assume they are very reflective thin discs and that all of the mass is in the sail. What is the maximum area density so that the sail does not fall towards the sun? Does the distance matter (find the general density to distance equation if it does matter)?

(b) **(3 pts)** Assume the solar sails are a thin spherical shell and made of a perfectly absorbent material instead. The area density is exactly half the maximum area density so that it does not fall towards the sun. What is its final speed if it starts at a stationary position 1 Au from the sun?

(c) **(5 pts)** Assume the solar sails are a thin spherical shell and made of material having reflectance r. With the same area density as in part (b), what is the final speed (it can reach this speed either when crashing into the sun, escaping the solar system, or remain in orbit)? This time, assume it starts in circular orbit 1 Au away from the sun as the starting condition. What is the orbital shape?

(d) **(6 pts)** Simply not falling into the sun is insufficient for solar sails. Most are planned to reach relativistic speeds. A way to achieve this is to fire large Earthbound lasers at the sail. How powerful must the lasers be to accelerate the sail up to $v = 0.2c$ in 50 days? Since this far exceeds the value obtained in part 3, neglect the effects of the sun.

(e) **(10 pts)** Suppose a perfectly reflecting solar sail in the shape of a thin disk (with mass m and radius r) orbiting around the star in a circular orbit of radius $d \gg r$. It is also spinning around itself, such that the spin angular velocity is in the same direction as the orbital angular velocity, and its axis of symmetry always remains parallel to the plane of the orbit. If its initial spin velocity is ω_0 find its spin velocity after it revolves an angle θ around the star.

Solution 2:

(a) Let the sail have an area of A and area density σ and the sun have mass M. The gravitational force acting on this sail is:

$$F_G = \frac{GM(A\sigma)}{R^2}$$

The photons emitted from the sun have a momentum of $\frac{E}{c}$. On the reflective sail, the change in momentum is:

$$\frac{2E}{c}$$

The force imparted from this change is:

$$F = \frac{dp}{dt} = \frac{2}{c}\frac{dE}{dt}$$

Now taking the projection area of the disk which is A in this case, we can find the fraction of the sun's power (denoted by P) reaching the sail:

$$\frac{PA}{4\pi R^2}$$

So the force from the photons emitted by the sun is

$$F_P = \frac{PA}{2\pi c R^2}$$

Now force balancing and solving algebraically we get,

$$\sigma = \frac{P}{2\pi G M c}$$

(b) The change in momentum for the photons is reduced by a factor of two because of the new perfectly absorbing sails:

$$F = \frac{1}{c}\frac{dE}{dt}$$

If we let the spherical shell have a surface area of A, the projection area of the spherical shell is $\frac{1}{4}A$. Using the same analysis in part a we find that

$$\sigma_{\text{max}} = \frac{PA}{16\pi c R^2}$$

and

$$F_P = \frac{PA}{16\pi c R^2}$$

Now using $\frac{1}{2}\sigma_{max}$ for our force balance equation we get,

$$(A\sigma)\ddot{r} = F_P - F_G$$

$$(A\sigma)\ddot{r} = -\frac{PA}{32\pi c R^2}$$

We find that the acceleration is $\frac{GM}{R^2}$ which means the magnitude of the gravitational potential field is $\frac{GM}{R}$. Now using conservation of energy (the initial kinetic energy and the final potential energy are zero):

$$\frac{1}{2}mv_f^2 = \frac{GMm}{R}$$

$$v_f = \sqrt{\frac{2GM}{R}}$$

(c) The absorbance is $1 - r$, so the force contribution from absorbed photons is:

$$F_{P1} = (1-r)\frac{PA}{4\pi c R^2}$$

where A is the area of projection of the sphere. For reflected photons, we only consider the change of momentum in the same direction of the original travelling direction since any impulse in the perpendicular directions cancel due to spherical symmetry. Incoming photons that are reflected at an incident angle θ have a change in momentum of

$$\frac{E}{c}(1+\cos(2\theta))$$

in the direction of travel. This is better illustrated in the following diagram:

It is important to note that at different angles there are varying frequencies of photons reflecting off at that angle due to a varying area of projection. It is easy to show that the area of projection at an angle θ is

$$2\pi r^2 \sin(\theta)\cos(\theta)\mathrm{d}\theta \ = A\sin(2\theta)\mathrm{d}\theta.$$

Taking the sum over all contributions from rings at all angles, we get the expression

$$F_{P2} = r\frac{PA}{4\pi cR^2}\int_0^{\pi/2}(1+2\cos(2\theta))(\sin(2\theta))\mathrm{d}\theta$$

After evaluating, we get that

$$F_{P2} = r\frac{PA}{4\pi cR^2}$$

We can conclude that the force is independent of the reflectance value r because the force from absorbance and reflectance contributions sum to $\frac{PA}{4\pi cR^2}$.

Now to solve for v, we can use conservation of energy (similar to part b, except this time we have an initial kinetic energy as well) to get:

$$\frac{1}{2}mv_f^2 = \frac{GMm}{R} + \frac{1}{2}mv_o^2$$

$$v_o^2 = \frac{GM}{R}$$

$$v_f = \sqrt{\frac{3GM}{R}}$$

It is well known that the eccentricity of an orbit in an inverse square relationship with two masses is:

$$e = \sqrt{1 + \frac{2L^2E}{m_r k^2}}$$

With E being the total energy, L the angular momentum, m_r the reduced mass, and k a constant. Because the energy is positive, the eccentricity is greater than one. Therefore it is a hyperbolic orbit. (Note: points for describing the orbit were given to teams that went through the complete derivation of the eccentricity or at least provided some sort of mathematical analysis)

(d) We need to take into account both relativistic effects as well as the doppler effect. The force of radiation is given by

$$F_{\text{rad}} = \frac{2P}{c}\sqrt{\frac{1-\beta}{1+\beta}}$$

which gives rise to the net force

$$\gamma^3 mc\frac{d\beta}{dt} = \frac{2P}{c}\sqrt{\frac{1-\beta}{1+\beta}}.$$

Solving this differential equation gives us

$$P = \frac{0.113mc^2}{t}.$$

(e) Assume that the intensity of the solar radiation at the solar sail is I. Let the solar sail have mass m uniformly distributed across a disc of radius r, and be perfectly reflecting. Suppose it is rotating with angular speed ω, about an axis on its plane (so by perpendicular axis theorem, it has moment of inertia $\frac{1}{4}mr^2$ about that axis). We break the solar sail into many tiny pieces of area dA and analyze the momentum transferred to each piece by the solar radiation.

Suppose a piece of area dA has its normal oriented an angle θ from the oncoming radiation, and it is moving a speed v directly along its normal. Consider what happens after a small time dt. Assume that the total momentum of the photons it intercepts is dp. The total momentum imparted to the area by those photons can be calculated by doppler shift, resulting in $2(\cos\theta + \frac{v}{c})dp$. Next, drawing a picture will show that the total momentum of the photons it intercepts is $(1 + \frac{v}{c\cos\theta})\frac{I}{c}\cos\theta dAdt$.

Thus, we know $2(\cos\theta + \frac{v}{c})(1 + \frac{v}{c\cos\theta})\frac{I}{c}\cos\theta dAdt$ gives the total momentum imparted to the piece of area by the radiation.

Now, we integrate all those small pieces of area of find the total angular impulse in time dt. Let α be the angular position of the small piece of area with respect to the axis of rotation of the solar sail. Then we get

$$\tau = \int_0^\pi 2(\cos\theta + \frac{\omega r\sin\alpha}{c})(1 + \frac{\omega r\sin\alpha}{c\cos\theta})\frac{I}{c}\cos\theta * 2(r\cos\alpha)^2 r\sin\alpha d\alpha -$$

$$\int_0^\pi 2(\cos\theta - \frac{\omega r\sin\alpha}{c})(1 - \frac{\omega r\sin\alpha}{c\cos\theta})\frac{I}{c}\cos\theta * 2(r\cos\alpha)^2 r\sin\alpha d\alpha$$

where τ is the angular impulse delivered per unit time. Note that the first integral represents the side of the disc that moves toward the sun, while the second integral represents the side of the disc that moves away from the sun. Also, notice we replaced v with $\omega r\sin\alpha$ in the first integral and v with $-\omega r\sin\alpha$ in the second integral. We also included the total area of the small pieces that were a distance $r\sin\alpha$ from the axis of rotation, which is $2(r\cos\alpha)^2 d\alpha$, and the lever arm is $r\sin\alpha$. After simplification, we get

$$\tau = 16\frac{I}{c^2}\omega r^4\cos\theta\int_0^\pi \cos^2\alpha\sin^2\alpha d\alpha$$

Finally, we get the average value of $|\tau|$ over θ is $\tau = \frac{4}{c^2}I\omega r^4 = -\frac{1}{4}mr^2\dot{\omega}$, since the torque is against the direction of rotation. Solving the differential equation gives $\omega(t) = \omega_0 e^{-\frac{16Ir^2}{mc^2}t}$.

Problem 3: Electron Escape (28 pts)

An infinite wire with current I has a radius a. The wire is made out of a material with resistivity ρ and heat conductivity κ. The temperature outside the wire is a constant T_0.

(a) **(4 pts)** After a long time, determine the temperature $T(r)$ at a distance r from the center of the wire. Assume that the current in the wire is uniformly distributed.

(b) **(4 pts)** Now, the outer surface of the wire is maintained at a potential of $-V$, where V is positive. The wire is surrounded by an infinite cylindrical shell with radius $b > a$ that is grounded. Somehow, an electron is able to escape from the wire. Assume that it is at rest just as it escapes. You can neglect radiation from the electron.

Draw a qualitative graph of the physical path that the electron takes, along with a diagram of the wire.

(c) **(6 pts)** Find the maximum distance r_{max} of the electron from the center of the wire in the subsequent motion as a function of V, and also in terms of I, a, and b. Ignore all relativistic effects in this part only.

(d) **(12 pts)** Redo the calculation in the previous part with relativistic effects.

(e) **(2 pts)** Graph the maximum radius r_{max} according to part (d) as a function of V.

Solution 3:

(a) As the current density is uniform, the current through a concentric circle with radius r will have total current $\frac{r^2}{a^2}I$. The resistance of the portion of the wire of length ℓ up to a radius r is given by

$$R = \frac{\rho\ell}{A} = \frac{\rho\ell}{\pi r^2}.$$

Therefore, the power emitted out of a cylindrical shell with radius r and length ℓ is

$$P = I^2R = \frac{r^4}{a^4}I^2\frac{\rho\ell}{\pi r^2} = \frac{\rho I^2\ell r^2}{\pi a^4}.$$

By Fourier's Law,

$$\frac{\rho I^2\ell r^2}{\pi a^4} = -\kappa A\frac{dT}{dr} = -\kappa(2\pi r\ell)\frac{dT}{dr}.$$

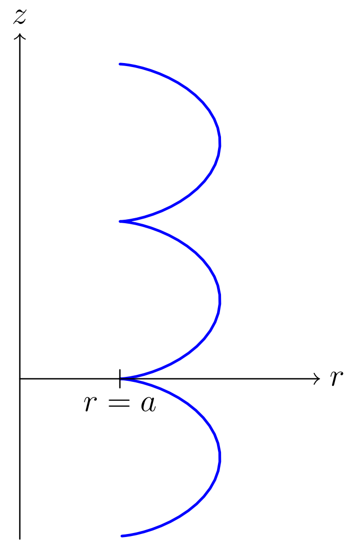
Therefore,

$$\frac{dT}{dr} = -\frac{\rho I^2}{2\pi^2a^4}r \implies T(r) = -\frac{\rho I^2}{4\pi^2a^4}r^2 + C.$$

Since $T(a) = T_0$, so we see $C = \frac{\rho I^2}{4\pi^2a^4}\cdot a^2 + T_0$. Thus,

$$\boxed{T(r) = T_0 + \frac{\rho I^2}{4\pi^2a^4}(a^2 - r^2)}.$$

(b) The electric force is constantly outward, while the magnetic force is perpendicular to the motion in the plane determined by the z axis and the radial direction. Note that the motion of the electron is always in this plane. Therefore, the electron will move outward and continuously change direction until it reaches a maximum radius. Then, it will move back towards the wire in a motion that is completely symmetric. When it reaches the wire again, the motion will repeat in a somewhat "cycloidal" motion. See the graph below.



(c) Clearly, there is only movement in the radial and z directions. The magnetic field at a radius r is given by $\frac{\mu_0 I}{2\pi r}$, and it is tangential. From Newton's second law in the z direction,

$$m\frac{dv_z}{dt} = e\dot{r}\left(\frac{\mu_0 I}{2\pi r}\right) \implies v_z = \frac{\mu_0 e I}{2\pi m}\ln\left(\frac{r}{a}\right).$$

By energy conservation, the energy of the electron at a distance r from the z-axis is given by

$$E = \frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right) = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_z^2.$$

At the maximum radius, $v_r = 0$, so we get

$$\frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r_{\max}}{a}\right) = \frac{1}{2}mv_z^2 = \frac{\mu_0^2 e^2 I^2}{8\pi^2 m}\left(\ln\frac{r_{\max}}{a}\right)^2.$$

Solving for $r_{\max}$,

$$\ln\left(\frac{r_{\max}}{a}\right) = \frac{8\pi^2 mV}{\mu_0^2 e I^2 \ln\left(\frac{b}{a}\right)} \implies \boxed{r_{\max} = a\exp\left(\frac{8\pi^2 mV}{\mu_0^2 e I^2\ln\left(\frac{b}{a}\right)}\right)}.$$

Noe that when this expression is greater than b, the electron stops at a radius b.

(d) Clearly, there is only movement in the radial and z directions. From Newton's second law in the z direction,

$$\frac{dp_z}{dt} = e\dot{r}\left(\frac{\mu_0 I}{2\pi r}\right) \implies p_z = \frac{\mu_0 e I}{2\pi}\ln\left(\frac{r}{a}\right).$$

Also, note by energy conservation that the energy of the electron at a distance r from the z-axis is

$$E = mc^2 + \frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right).$$

Therefore,we have,

$$E^2 = (mc^2)^2 + (pc)^2 = (mc^2)^2 + (p_r^2 + p_z^2)c^2 = \left(mc^2 + \frac{eV}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right)\right)^2,$$

$$p_r^2 + p_z^2 = \frac{e^2V^2\ln\left(\frac{r}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)},$$

$$p_r^2 = \frac{e^2V^2\ln\left(\frac{r}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)} - \frac{\mu_0^2e^2I^2}{4\pi^2}\left(\ln\left(\frac{r}{a}\right)\right)^2$$

At the maximum radius, r_{max}, we have $p_r = 0$. Therefore,

$$\frac{e^2V^2\ln\left(\frac{r_{\text{max}}}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV\ln\left(\frac{r_{\text{max}}}{a}\right)}{\ln\left(\frac{b}{a}\right)} - \frac{\mu_0^2e^2I^2}{4\pi^2}\left(\ln\left(\frac{r_{\text{max}}}{a}\right)\right)^2 = 0.$$

We can solve for r_{max} to get

$$r_{\text{max}} = a\exp\left(\frac{2meV}{\ln\left(\frac{b}{a}\right)}\left(\frac{\mu_0^2e^2I^2}{4\pi^2} - \frac{e^2V^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2}\right)^{-1}\right).$$

However, we also have to note that $r_{\text{max}} \leq b$. We will now find the condition on V such that the electron's maximum radius is b. The critical voltage V_{crit} is when $p_r = 0$ at $r = b$. We have

$$\frac{e^2V_{\text{crit}}^2\ln\left(\frac{b}{a}\right)^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2meV_{\text{crit}}\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{a}\right)} - \frac{\mu_0^2e^2I^2}{4\pi^2}\left(\ln\left(\frac{b}{a}\right)\right)^2 = 0,$$

$$e^2V_{\text{crit}}^2 + 2mc^2eV_{\text{crit}} - \frac{\mu_0^2e^2I^2c^2}{4\pi^2}\left(\ln\left(\frac{b}{a}\right)\right)^2 = 0.$$

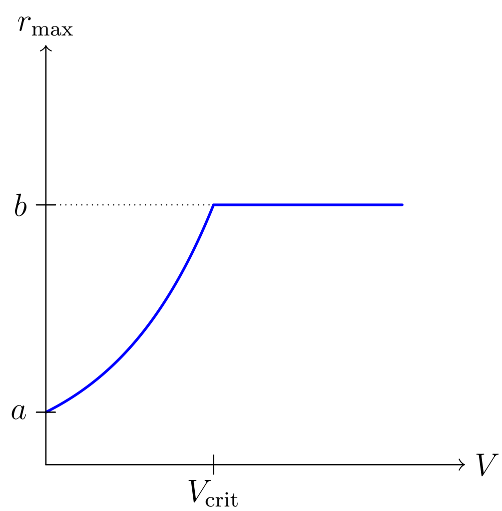
By the quadratic formula,

$$\boxed{V_{\text{crit}} = \frac{\sqrt{4m^2c^4e^2 + \frac{\mu_0^2e^4I^2c^2}{\pi^2}\left(\ln\frac{b}{a}\right)^2} - 2mc^2e}{2e^2}}.$$

Therefore, we have

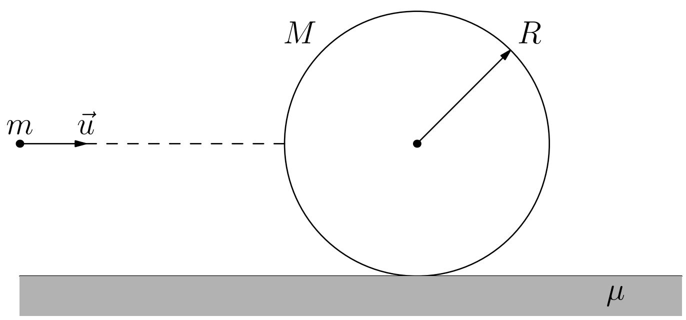
$$\boxed{r_{\text{max}} = \begin{cases} a\exp\left(\frac{2meV}{\ln\left(\frac{b}{a}\right)}\left(\frac{\mu_0^2e^2I^2}{4\pi^2} - \frac{e^2V^2}{c^2\left(\ln\left(\frac{b}{a}\right)\right)^2}\right)^{-1}\right) & 0 \leq V < V_{\text{crit}} \\ b & V \geq V_{\text{crit}} \end{cases}}.$$

(e) The graph is roughly exponential in V until $r_{\text{max}} = b$, at which point it is flat.



Problem 4: Bullet in Cylinder (20 pts)

A hollow cylinder of mass M and radius R rests on a rough horizontal surface. A projectile of mass $m < M$ having a velocity u directed horizontally exactly towards the middle of the cylinder as shown in the figure. The shell gets stuck in the cylinder wall, after which the shell starts to move, slipping on the surface. The coefficient of static and kinetic friction between the cylinder and the horizontal surface are the same, and are equal to $\mu < 2$.



(a) **(4 pt)** In which direction does the cylinder rotate? State your answers for different values of μ.

(b) **(6 pts)** Find its angular acceleration about the centre of the cylinder just after the impact.

(c) **(10 pts)** It is known that some time after the impact, the horizontal projection of the velocity of the center of mass of the system is equal to v, and the angular velocity of the cylinder is Ω. Till this point, the cylinder has rotated through an angle ϕ.

How much heat was released in the system till this point, if the cylinder all the time after the impact moved with slipping, rotating in one direction? Assume that all energy losses are dissipated in the form of heat.

Solution 4:

(a) Combining both force equations in the vertical direction and our 'torque' equation and that $F_0 = \mu N$, we get

$$\alpha = \frac{g}{R} \cdot \frac{(\mu - 1)m + \mu M}{(1 - \mu)m + M}$$

$$N = \frac{(2m + M)Mg}{M + (1 - \mu)m}$$

Note that α and N tend to infinity as $\mu \to \mu_0 = 1 + \frac{M}{m}$. This means that the normal force can even increase infinitely if a large impulse is provided by the bullet. For $m = M$, $\mu_0 = 2$, and since the problem says that $\mu < 2$, we know that both N and α are finite and always remain true.

Now, for the direction of rotation, let us observe when α changes parity. Notice that for $\mu < \mu_1 = \frac{m}{M+m}$, we have $\alpha < 0$, meaning that the cylinder will rotate counter-clockwise. Let us try to understand what could've caused such a situation? This could've been caused by a large impulse by the bullet, when it exceeded the impulse due to friction. Similarly, for $\mu > \mu_1$, we have that the cylinder rotates clockwise. Note that since $m < M$, $0 < \mu_1 < \frac{1}{2}$. So are answers for the first part are

$$\text{Direction of rotation} = \begin{cases} \text{Clockwise} & \text{if } \mu > \frac{m}{M+m} \\ \text{Counter-clockwise} & \text{if } \mu < \frac{m}{M+m} \end{cases}$$

(b) Now we try to find the heat released in the system at any time. For the sake of a definite solution, assume that the cylinder has rotated an angle ϕ clockwise. At this moment, we have the velocity v of the cylinder along the horizontal, angular velocity Ω, and the linear velocity of rotation of the bullet v_1. (clearly $v_1 = \Omega R$) Now let us write the energy of the bullet in the reference frame of the axis of the cylinder:

$$E = mgR\sin\phi + \frac{1}{2}m{v_1}^2$$

Transcending back to the original lab frame, since we need to replace the velocity of the bullet accordingly, and using $v_1 = \Omega R$, we get

$$E = mgR\sin\phi + \frac{1}{2}m{v_0}^2 + \frac{1}{2}m\Omega^2R^2 + m\Omega R v_0 \sin\phi$$

and we have the energy of the cylinder in the lab frame

$$E_c = \frac{1}{2}M{v_0}^2 + \frac{1}{2}M\Omega^2R^2$$

Also note that from transitioning from the cylinder axis' frame to the lab frame, we convert the centre of mass velocity as

$$v = v_0 + \Omega r \sin\phi$$

where r is the distance of the cylinder's centre to the new centre of mass. The value of r can be found easily:

$$Mr = m(R - r) \Rightarrow r = \frac{m}{M+m}R$$

So the work-energy theorem becomes

$$\Delta H = E + E_c - \frac{1}{2}m{v_0}^2$$

Substituting the values of E and E_c, we finally get

$$\Delta H = \begin{cases} \frac{1}{2}mu^2 + \frac{m^2\Omega^2R^2\sin^2\phi}{2(m+M)} - \frac{m+M}{2}(v^2 + \Omega^2R^2) + mgR\sin\phi & \text{if counter-clockwise} \\ \frac{1}{2}mu^2 + \frac{m^2\Omega^2R^2\sin^2\phi}{2(m+M)} - \frac{m+M}{2}(v^2 + \Omega^2R^2) - mgR\sin\phi & \text{if clockwise} \end{cases}$$

Comment (not for grading purposes):

In the process, we assumed that the combined system of bullet and cylinder does not start its motion by "jumping up". This directly affects our approach to the problem and is not entirely obvious.
In the perfectly inelastic collision, the angular momentum of the bullet-cylinder system is conserved. As above, we assume the velocity of the centre of mass v to be directed along the horizontal.

Problem 5: Mathematical Physics (18 pts)

The study of mathematics has almost always paved the way for the development of new ideas in physics. Newtonian mechanics could not be possible without first inventing calculus, and general relativity could not have existed without heavy development in tensors. However, there are numerous cases where physical insight have paved the way for mathematics.

Perhaps the most notable would be the Brachistochrone problem, which asks for the path that leads to the fastest descent influenced by gravity between two given points. While it is solvable through the calculus of variations, Newton proposed an easier solution by modelling the path of light through a medium with a variable index of refraction. You may read about this problem and the fascinating history behind it here.

We will not be dealing with this specific problem, but rather multiple short mathematics problems that can be can be represented with a physical analog. To receive points, you must use the suggested physical set-up.

(a) **(4 pts)** Show that for small values of x, we have

$$\cos(x) = 1 - \frac{x^2}{2}.$$

Physical Setup: Consider a small object moving in a circle.

(b) **(8 pts)** A ladder of length ℓ with a thickness of 0.3 m is transported around a right angled corner where the two hallways leading up to it have a width of 3 m and 5 m. What is the maximum length of the ladder such that it can be successfully transported across?

Physical Setup: Consider the ladder as a compressed spring that can freely expand in only its longitudinal direction. Do not explicitly take derivatives. Instead, consider a force/torque balance. You may or may not need to solve an equation numerically.

(c) **(6 pts)** Prove the AM-QM inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

Physical Setup: Design circuit(s) and compare their measurable quantities with each other qualitatively (or otherwise).

Solution 5:

(a) Consider a small particle starting from rest on a circular path. It quickly and uniformly accelerates to a speed v in a very small time interval Δt such that $\frac{v\Delta t}{R} \ll 1$ where R is the radius of the circle.

Since this time interval is so small, the acceleration is directed inwards. Without loss of generality, let us define the inwards direction as the positive $+y$ direction. The displacement Δy is given by $R(1 - \cos\theta)$ which gives:

$$R(1 - \cos\theta) = \frac{1}{2}at^2$$

At time t, since the particle hasn't moved very far, the inwards acceleration is still pointed in the $+y$ direction and using equations for circular motion, it has a magnitude of:

$$a = \frac{v^2}{R}$$

Making the substituting, we can simplify:

$$1 - \cos\theta = \frac{1}{2}\left(\frac{vt}{R}\right)^2$$

Since $vt = R\theta$ is the distance traveled, this simplifies to:

$$1 - \cos\theta = \frac{1}{2}\theta^2 \implies \boxed{\cos\theta = 1 - \frac{1}{2}\theta^2}$$

(b) Treating the ladder as a spring, we see that its length is directly related to its potential energy. Determining the point at which the ladder's length is at a minimum is equivalent to determining the point at which the rigid spring's potential energy is at a minimum. Using the fact that:

$$F = -\frac{dU}{dx}$$

implies that when the potential energy is at a minimum, the net force must be equal to zero. Therefore, we can set up a force and torque balance in order to solve for the shortest length. There are three normal forces to take into account. Balancing forces gives two equations, and a torque balance gives the third. Solving a system of three equations will allow us to solve the problem.

Perhaps a slicker way is to realize that these three forces must be concurrent. At least two of the normal force vectors will intersect, and performing a torque balance where the pivot is selected to be that intersection point, these two normal force vectors will not contribute to a torque. In order to satisfy $\sum \tau = 0$, the third normal force vector must also intersect at this same place. This turns it into a geometry problem.

Solving the problem geometrically or analytically, you end up solving the equation:

$$\frac{a\cos\theta - \ell}{\sin^2\theta} = \frac{b\sin\theta - \ell}{\cos^2\theta}$$

which is possible to be solved numerically (or solved through an approximation). This wasn't specified on the exam, so most teams who arrived at this equation would have received the marks.

Solving, we get $L \approx 10.6$ m.

(c) We connect several batteries with an electromotive force ε_i and an identical internal resistance r in parallel with one another, and the entire system is in parallel with a wire with zero resistance. Using Kirchoff's Loop rule, the voltage drops across each internal resistor is going to be the same. Therefore, the power dissipation is:

$$P_1 = \frac{\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_N^2}{r}$$

if there are N such resistors. We then create a second circuit where all of these resistors are in series. The current is now the same and the total power dissipation is:

$$P_2 = \frac{(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)^2}{Nr}$$

If we make the claim that $P_1 \geq P_2$, the AM-QM inequality will naturally follow. To do this, we can look at the amp-hour capacity of batteries in parallel vs in series. In series, the amp-hour capacity of each resistor must be the same (and be equal to the capacity of the weakest battery). If this was not the case, then some batteries will be depleted before others. This is not the case for parallel and will only be the same if $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_N$, in which case we have $P_1 = P_2$. Due to this, the power dissipation in the first circuit must be equal or greater than the one in the second circuit, proving the AM-QM inequality.

Problem 6: Flat Earth (22 pts)

In this problem, we will explore the true gravitational model of the earth, not the one that is claimed in most textbooks. Contrary to popular belief, the Earth is a flat circle of radius R and has a uniform mass per unit area σ. The Earth rotates with angular velocity ω.

(a) **(5 pts)** A pendulum of length ℓ that is constrained to only move in one plane is placed on the ground at the center of the Earth. The pendulum has more than one angular frequency of small oscillations. Find the value of each angular frequency of small oscillations $\Omega(0), \Omega_1(0), ...$ in terms of σ, ω, ℓ, and physical constants and the equilibrium angle $\theta, \theta_1, ...$ that the frequency occurs at. Assume for all parts that $\ell \ll R$.

An equilibrium angle corresponds to the angle with respect to the vertical where there is an equilibrium point.

(b) **(2 pt)** Investigate the stability of each equilibrium position with varying angular velocity of the Earth.

(c) **(12 pts)** The entire pendulum is moved a horizontal distance $r \ll R$ away from the center of the Earth. It is oriented so that it is constrained to only move in the radial direction. Now, find the new angular frequency $\Omega(r)$ of small oscillations about the lowest equilibrium point in terms of the given parameters, assuming that $\omega^2 r$ is much less than the local gravitational acceleration.

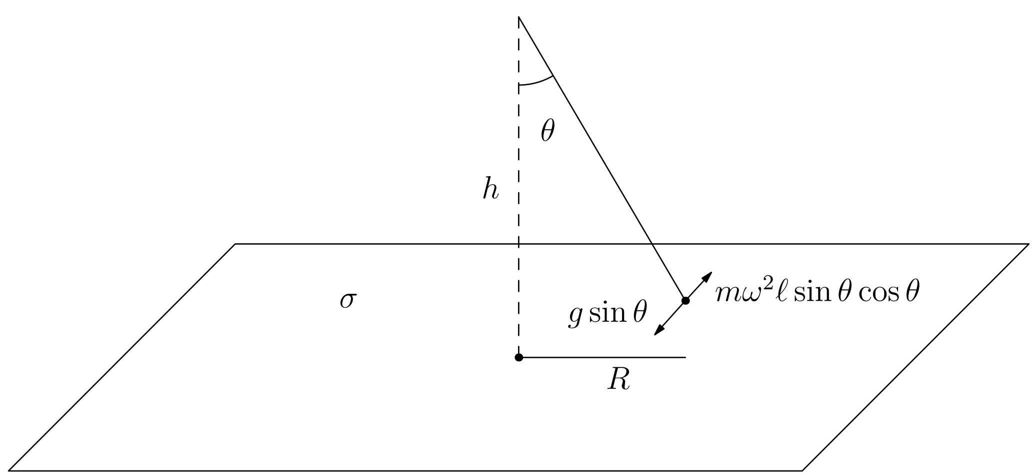
(d) **(3 pts)** The angular frequencies $\Omega(0)$ and $\Omega(r)$ are both measured and the difference is found to be $\Delta\Omega$. Assuming that $\Delta\Omega \ll \Omega(0)$ and $\omega^2 \ll \frac{g}{\ell}$, determine σ in terms of $\omega, r, \Omega(0), \Delta\Omega$, and physical constants.

Solution 6:

(a) The first thing that we note is that the acceleration due to the gravitational force is different as compared to the one on Earth. Consider a cylindrical Gaussian surface that has it's circular faces parallel to the Earth. From Gauss's Law of gravitation, we can calculate the acceleration due to gravity on a flat plane to be

$$4\pi G m = 2 \times \pi r^2 g \implies g = 2\pi G\sigma$$

by substituting $m = \sigma \pi r^2$. Let us now consider a small displacement of the mass m from the origin. We see that there are two forces involved: the centrifugal force, and the gravitational force. We now draw a free-body diagram as shown below:



Note that the gravitational force will be restoring and is given by $mg\sin\theta \approx mg\theta$. The centrifugal force is directed outwards and is given by $m\omega^2\ell\sin\theta\cos\theta \approx m\omega^2\ell\theta$. Writing Newton's Laws on the pendulum now gives us

$$m\ell\ddot{\theta} = m\omega^2\ell\theta - mg\theta \implies \ddot{\theta} = -\left(\frac{g}{\ell} - \omega^2\right)\theta.$$

An equilibrium position occurs when $\ddot{\theta} = 0$. This tells us that the two equilibrium positions are defined by

$$\theta_1 = 0, \qquad \theta_2 = \arccos\left(\frac{g}{\omega^2\ell}\right).$$

For oscillations near $\theta_1 = 0$, we can use small-angle approximations gives us

$$\ddot{\theta} = -\left(\frac{g}{\ell} - \omega^2\right)\theta.$$

Therefore, we see $\Omega(0) = \sqrt{\frac{g}{\ell} - \omega^2} = \boxed{\sqrt{\frac{2\pi G\sigma}{\ell} - \omega^2}}$.

For oscillations near $\theta_2 = \arccos\left(\frac{g}{\omega^2\ell}\right)$, we let $\theta = \theta_1 + \varphi$ where $\varphi \ll \theta_1$. Using a first order approximation tells us that

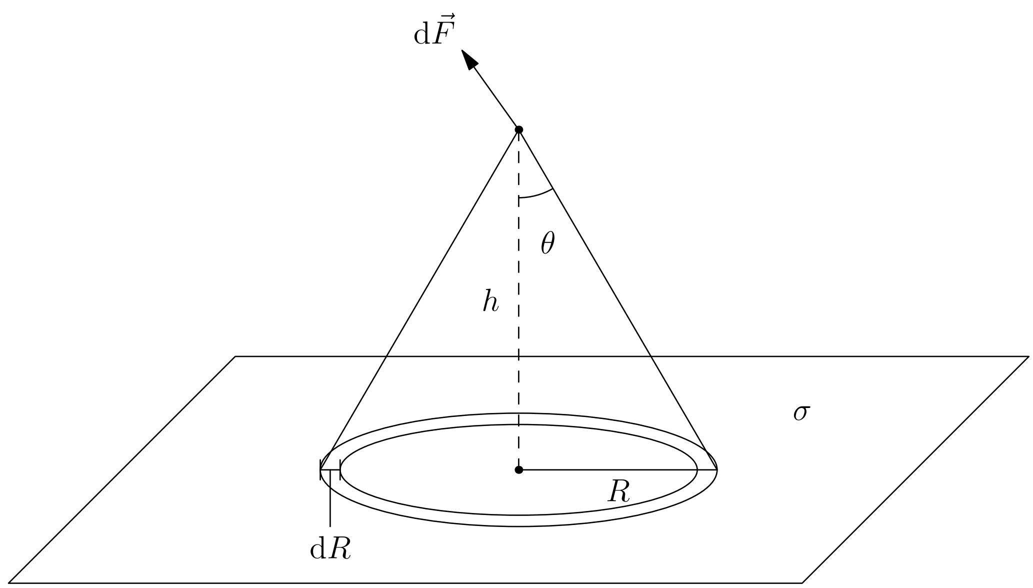
$$\ddot{\varphi} + \left(\frac{g}{\ell} + \omega^2\varphi\sin\theta_1 - \omega^2\cos\theta_1\right)(\sin\theta_1 + \varphi\cos\theta_1) = 0.$$

Simplifying gives us

$$\ddot{\varphi} = -(\omega^2\sin^2\theta_1)\varphi.$$

This tells us that $\Omega_1(0) = \omega\sin\theta = \omega\sqrt{1 - \frac{g^2}{\omega^4\ell^2}} = \boxed{\frac{1}{\omega\ell}\sqrt{\omega^4\ell^2 - 4\pi^2G^2\sigma^2}}$.

Remark 1: We can find the gravitational acceleration on the planet by in fact another way of analyzing the force per each infinitesimal ring. We can split the flat Earth into many tiny rings as shown below



Each ring provides a force of $d\vec{F}$. If each ring has a thickness dR, our force will be given by

$$dF = \frac{Gm\sigma}{h^2 + R^2}\cos\theta \cdot (\pi(R + dR)^2 - \pi R^2).$$

Using the fact that $\cos\theta = \frac{h}{\sqrt{h^2+R^2}}$, we then find that

$$dF = \frac{Gm\sigma}{h^2+R^2}\frac{h}{\sqrt{h^2+R^2}} \cdot (\pi(R+dR)^2 - \pi R^2).$$

Using a first order approximation gives us

$$(\pi(R+dR)^2 - \pi R^2) \approx 2RdR.$$

We then find by substituting that,

$$dF = \frac{2GmR\sigma h}{(h^2+R^2)^{3/2}}dR.$$

Now, integrating this force dF from 0 to ∞ gives us

$$F = \int_0^\infty dF = 2Gm\sigma h \int_0^\infty \frac{R}{(h^2+R^2)^{3/2}}dR = 2\pi G\sigma m.$$

The gravitational acceleration is then given by

$$g \equiv \frac{2\pi G\sigma m}{m} = 2\pi G\sigma.$$

■

Remark 2: We can find the equation of motion with the Euler-Lagrange equations. Let us consider a rotating frame at the center of the Earth. The pendulum's coordinates are then given by

$$(x, y, z) = (\ell\sin\theta, 0, -\ell\cos\theta)$$

which implies that the velocity of the pendulum is given by

$$(\dot{x}, \dot{y}, \dot{z}) = (\ell\dot{\theta}\cos\theta, 0, \ell\dot{\theta}\sin\theta).$$

In the fixed frame, the pendulum has an additional velocity from the centrifugal force which is given by

$$\vec{\omega} \times \vec{r} = (0, 0, \omega) \times (\ell\sin\theta, 0, -\ell\cos\theta) = (0, \omega\ell\sin\theta, 0).$$

We now write the lagrangian of the system as

$$\begin{aligned}\mathcal{L} &\equiv T - V \\ &= \frac{1}{2}m\sqrt{(\ell\dot{\theta}\cos\theta)^2 + 0^2 + (\ell\dot{\theta}\sin\theta)^2} + \frac{1}{2}m\ell^2\omega^2\sin^2\theta + mg\ell\cos\theta \\ &= \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\omega^2\sin^2\theta + mg\ell\cos\theta.\end{aligned}$$

Now, using the Euler-Lagrange equations gives us

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \implies \ddot{\theta} = -\left(\frac{g}{\ell} - \omega^2\cos\theta\right)\sin\theta.$$

■

(b) We have two cases:

Case 1: (oscillations for $\Omega(0)$) If $\omega < \sqrt{\frac{g}{\ell}} = \sqrt{\frac{2\pi G\sigma}{\ell}}$, we see that the oscillations are stable. The oscillations would follow the equation

$$\theta(t) = A_1\cos(\Omega(0) + \phi_1)$$

where A_1 and ϕ_1 are constants to be determined from initial conditions.

Case 2: (oscillations for $\Omega_1(0)$) If $\omega > \sqrt{\frac{2\pi G\sigma}{\ell}}$ we see that the oscillations are stable. The oscillations would follow the equation

$$\varphi(t) = A_2 \cos(\Omega_1(0) + \phi_2)$$

where A_2 and ϕ_2 are constants to be determined from initial conditions.

If $\omega = \sqrt{\frac{2\pi G\sigma}{\ell}}$ the oscillations are neutrally stable but do not display stable small oscillations.

(c) We use the fact that, for small oscillations of an object subject to a potential $U(x)$, the frequency of small oscillations will be defined as

$$\omega = \sqrt{\frac{U''(x)}{m_{\text{eff}}}}.$$

The potential energy is due to the gravitational potential and the centrifugal potential:

$$U = -mg\ell\cos\theta - \frac{1}{2}m\omega^2(r + \ell\sin\theta)^2$$

We see that

$$\frac{d^2U}{d\theta^2} = mg\ell\cos\theta + m\omega^2 r\ell\sin\theta + m\omega^2\ell^2(\sin^2\theta - \cos^2\theta)$$

When evaluated at $\theta = \theta_0$, the equilibrium position,

$$U''(\theta_0) = mg\ell\cos\theta_0 + m\omega^2 r\ell\sin\theta_0 + m\omega^2\ell^2(\sin^2\theta_0 - \cos^2\theta_0) = k_{\text{eff}}.$$

We now need to find the effective mass. Note that the kinetic energy is given by

$$\frac{1}{2}m\ell^2\dot{\theta}^2 = \frac{1}{2}m_{\text{eff}}\dot{\theta}^2 \implies m_{\text{eff}} = m\ell^2.$$

Thus, the frequency is given by

$$\Omega(r) = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{g}{\ell}\cos\theta_0 + \frac{\omega^2 r}{\ell}\sin\theta_0 + \omega^2(\sin^2\theta_0 - \cos^2\theta_0)}$$

Now, we have $\cos\theta_0 = \frac{g}{\sqrt{\omega^4 r^2 + g^2}} \approx 1 - \frac{\omega^4 r^2}{2g^2}$ and $\sin\theta_0 \approx \frac{\omega^2 r}{g}$. Substituting,

$$\Omega(r) = \sqrt{\frac{g}{\ell} - \frac{\omega^4 r^2}{2g\ell} + \frac{\omega^4 r^2}{g\ell} + \frac{\omega^6 r^2}{g^2} - \omega^2 + \frac{\omega^6 r^2}{2g^2}}$$

$$\Omega(r) = \sqrt{\Omega(0)^2 + \frac{\omega^4 r^2}{2g\ell} + \frac{3\omega^6 r^2}{2g^2}} = \boxed{\sqrt{\Omega(0)^2 + \frac{\omega^4 r^2}{4\pi G\sigma\ell} + \frac{3\omega^6 r^2}{8\pi^2 G^2\sigma^2}}}$$

(d) Note that we can write

$$\Omega(r) = \sqrt{\Omega(0)^2 + \frac{\omega^4 r^2}{2g^2}\left(\frac{g}{\ell} + 3\omega^2\right)} = \Omega(0)\sqrt{1 + \frac{\omega^4 r^2}{2g^2}\left(\frac{\frac{g}{\ell} + 3\omega^2}{\frac{g}{\ell} - \omega^2}\right)}$$

Therefore, we have

$$\Omega(r) \approx \Omega(0)\sqrt{1 + \frac{\omega^4 r^2}{2g^2}} \approx \Omega(0)\left(1 + \frac{\omega^4 r^2}{4g^2}\right)$$

Therefore, $\Delta\Omega = \frac{\omega^4 r^2}{4g^2}\Omega(0)$. This implies that

$$g^2 = 4\pi^2 G^2\sigma^2 = \frac{\omega^4 r^2\Omega(0)}{4\Delta\Omega}$$

We can solve for σ to get $\sigma = \boxed{\frac{\omega^2 r}{4\pi G}\sqrt{\frac{\Omega(0)}{\Delta\Omega}}}$.

Problem 7: Boltzmann Statistics (24 pts)

In this problem, we will explore Boltzmann Statistics and using it to build similar models for quantum particles such as bosons and fermions.

(a) (8 pts) Consider a energy of a gaseous molecule in space is given by 5

$$E = E_0(|x|^r + |y|^r + |z|^r)$$

where the coordinates of the molecule are represented by (x, y, z), E_0 is a constant with appropriate units, and r is a non-negative real number. The system is in thermal equilibrium with a reservoir of temperature T. Calculate explicitly using appropriate statistical methods, the average energy of a thermodynamic system consisting of such gaseous molecules, considering the Maxwell-Boltzmann distribution. Analyse your result and provide a qualitative argument to support it.

Hint: If you are not familiar with how to solve this part, try part (b) first.

(b) (6 pts) Consider a specific case in which

$$E = E_0(x^2 + y^2 + z^2)$$

except where $|x|, |y|, |z| < 2$ and particles can only exist at integer values of x, y, z. If the system is still in thermal equilibrium at a temperature T, calculate the average energy.

(c) (6 pts) In this system, there are two particles. What is the probability that at least one of these particles will be in the ground state (energy is zero) if:

(i) The two particles are distinct.

(ii) The two particles are identical bosons.

(iii) The two particles are identical fermions (fermions follow Pauli exclusion principle). For simplicity, ignore spin.

For each of the parts, assume that there are no other interactions (e.g. electromagnetism) and focus mainly on a statistical argument.

(d) (4 pts) Rank the probabilities in part (c) from highest to lowest when the temperature is

(i) high

(ii) low

For each, explain qualitatively why this must be true.

Solution 7:

(a) In Maxwell-Boltzmann's distribution, the probability that an isolated particle occupies the point $A(x, y, z)$ is given by

$$\mathrm{d}\mathbb{P}(E) = \mathcal{P}(E)\mathrm{d}V = -\frac{e^{-\frac{E_0}{k_B T}}}{Z}$$

where Z denotes the partition function. Note that although we write the probability such that an isolated particle occupied the space, but for a system of N particles, there would be an extra factor of $N!$ in the partition function, but since the particles are considered independent, this factor would cancel out in further calculation anyway. Now for the calculation of Z, we normalize the probability $\mathrm{d}\mathbb{P}(E)$ over all possible states; meaning that we convert the probability function to the probability density function by summing total probability as 1. This gives us

$$Z = \iiint_{\mathbb{R}^3} e^{-\beta \cdot E_0 \cdot (|x|^r + |y|^r + |z|^r)} \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

(More accurately, the partition function of a set of molecules would be found by summing, and not integrating over the space, but since it is a continuous function, the sum reduces to an integral.) From the statistical definition of average energy per unit particle, we have

$$\langle E \rangle = \iiint_{\mathbb{R}^3} E \cdot \mathrm{d}\mathbb{P}(E) = \iiint_{\mathbb{R}^3} E_0(|x|^r + |y|^r + |z|^r)\mathrm{d}\mathbb{P}(E)$$

Substituting the partition function from above, we obtain

$$\langle E\rangle = \frac{\iiint_{\mathbb{R}^3} E_0(|x|^r+|y|^r+|z|^r)e^{\left(-\frac{E_0}{k_BT}(|x|^r+|y|^r+|z|^r)\right)}\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_{\mathbb{R}^3} e^{\left(-\frac{E_0}{k_BT}(|x|^r+|y|^r+|z|^r)\right)}\mathrm{d}x\mathrm{d}y\mathrm{d}z}$$

$$\langle E\rangle = \frac{\sum_{x_i\in(x,y,z)}\iiint_{\mathbb{R}^3} e^{-\frac{E_0{x_i}^r}{kT}}E_0{x_i}^r\mathrm{d}x_i}{\sum_{x_i\in(x,y,z)}\iiint_{\mathbb{R}^3} e^{-\frac{E_0{x_i}^r}{k_BT}}\mathrm{d}x_i}$$

Now let us evaluate some integrals to obtain a closed form for $\langle E\rangle$ above. Applying by-parts, we have

$$\int_0^\infty e^{-\frac{E_0{x_i}^r}{k_BT}}\mathrm{d}x_i = \int_0^\infty e^{-\left(\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}x_i\right)^r}\mathrm{d}x_i$$

$$= x_i\left(\frac{E_0}{K_BT}\right)^{1/r}e^{-\frac{E_0x_i^r}{r_BT}}\Bigg|_0^\infty + r\int_0^\infty \frac{E_0x_i^r}{K_BT}e^{-\frac{E_0x_i^r}{k_BT}}\mathrm{d}\left(x_i\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}\right)$$

Note that after applying the bounds, $x_i\left(\frac{E_0}{K_BT}\right)^{1/r}e^{-\frac{E_0x_i^r}{r_BT}}\Big|_0^\infty$ reduces to 0. This gives

$$\int_0^\infty e^{-\left(\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}x_i\right)^r}\mathrm{d}x_i = r\int_0^\infty \frac{E_0x_i^r}{K_BT}e^{-\frac{E_0x_i^r}{k_BT}}\mathrm{d}\left(x_i\left(\frac{E_0}{k_BT}\right)^{\frac{1}{r}}\right)$$

The same integral ratio appeared in our original average energy per unit particle expression. However, the above integral was performed across one-dimension. In three dimensions (the integral is non-degenerate across all three dimensions), our average energy per particle expression becomes

$$\langle E\rangle = \frac{\iiint_{\mathbb{R}^3} e^{-\frac{E_0{x_i}^r}{kT}}E_0{x_i}^r\mathrm{d}x_i}{\sum_{x_i\in(x,y,z)}\iiint_{\mathbb{R}^3} e^{-\frac{E_0{x_i}^r}{k_BT}}\mathrm{d}x_i} = E_0\times\frac{3k_BT}{rE_0} = \frac{3k_BT}{r}$$

Hence, the average energy of a thermodynamic system of N such particles is

$$\langle E_s\rangle = \boxed{\frac{3Nk_BT}{r}}$$

As a sanity check, note that the average energy is independent of E_0, which was expected. This is because for any quadratic function in space (specifically, here this is $r=2$) we have the "Law of Equipartition", which states that every degree of freedom has an average energy of $1/2k_BT$, which is confirmed by the obtained expression.

Aliter: Consider the partition function:

$$Z = \sum e^{-\beta E} = \sum e^{-\beta E_0(x^r+y^r+z^r)}$$

We can turn this into the integral form:

$$Z = \frac{1}{\delta x\delta y\delta z}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\beta E_0(x^r+y^r+z^r)}\,dx\,dy\,dz$$

Letting $a=(\beta E_0)^{1/r}x$, $b=(\beta E_0)^{1/r}y$, and $c=(\beta E_0)^{1/r}z$, we can rewrite this as:

$$Z = \frac{1}{\delta x\delta y\delta z(\beta E_0)^{3/r}}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(a^r+b^r+c^r)}\,da\,db\,dc = \frac{C}{\delta x\delta y\delta z(\beta E_0)^{3/r}}$$

where C is a constant, equal to the integral. The average energy is given by:

$$\begin{aligned} E_{\text{avg}} &= \frac{-1}{Z}\frac{\partial Z}{\partial \beta} \\ &= \frac{-\delta x \delta y \delta z (\beta E_0)^{3/r}}{C} \frac{C}{\delta x \delta y \delta z}\left(\frac{-3}{r} E_0 (\beta E_0)^{\frac{-3-r}{r}}\right) \\ &= \frac{3}{r\beta} \\ &= \frac{3}{r} k_B T \end{aligned}$$

(b) The energy of any given particle can be $0, 1, 2, 3$ (working in units where $E_0 = 1$). We need to calculate the Boltzmann factor for each energy as well as how many ways each energy can be achieved.

- Energy 0: $e^{0\beta} = 1 \to 1$ way.
- Energy 1: $e^{1\beta} \to 6$ ways.
- Energy 2: $e^{2\beta} \to 12$ ways.
- Energy 3: $e^{3\beta} \to 8$ ways.

In total, the partition function is:

$$Z = 1 + 6e^{\beta} + 12e^{2\beta} + 8e^{3\beta}$$

The average energy is thus:

$$\begin{aligned} E_{\text{avg}} &= -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \\ &= -\frac{6e^{-\beta} + 24e^{-2\beta} + 24e^{-3\beta}}{1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}} \end{aligned}$$

Note that this can be factored further.

(c) The probability of at least one particle being in the ground state is equal to one minus the probability of no particles being in the ground state. The probability of having some energy E_i is:

$$P_i = \frac{(\text{number of ways this can happen})e^{-\beta E_i}}{Z}$$

Summing up all the probabilities for nonzero energies gives us:

$$P_{\text{at least 1 in ground state}} = 1 - \frac{Z_{\text{restrictions}}}{Z_{\text{no restrictions}}}$$

(i) If two particles are distinct, they can take on energies $1, 2, 3, 4, 5, 6$. Similar to part (b), we list out all the possible ways to achieve these energies if there are no restrictions:

- Energy 0: $1^2 = 1$ way.
- Energy 1: $2(1 \cdot 6) = 12$ ways.
- Energy 2: $2(1 \cdot 12) + 6^2 = 60$ ways.
- Energy 3: $2(1 \cdot 8) + 2(6 \cdot 12) = 160$ ways.
- Energy 4: $2(6 \cdot 8) + 12^2 = 240$ ways.
- Energy 5: $2(12 \cdot 8) = 192$ ways.
- Energy 6: $8^2 = 64$ ways.

If no particles are in the ground state, then:

- Energy 2: $6^2 = 36$ ways.
- Energy 3: $2(6 \cdot 12) = 144$ ways.
- Energy 4: $2(6 \cdot 8) + 12^2 = 240$ ways.
- Energy 5: $2(12 \cdot 8) = 192$ ways.
- Energy 6: $8^2 = 64$ ways.

The probability is thus:

$$P_1 = 1 - \frac{36e^{-2\beta} + 144e^{-3\beta} + 240e^{-4\beta} + 192e^{-5\beta} + 64e^{-6\beta}}{1 + 12e^{-\beta} + 60e^{-2\beta} + 160e^{-3\beta} + 240e^{-4\beta} + 192e^{-5\beta} + 64e^{-6\beta}}$$

Perhaps an easier way to do this problem is to think of the partition functions as generating functions. The partition function for placing two particles with no restrictions is the square of placing a single particle:

$$Z_{2,\text{ no restriction}} = Z^2_{1,\text{ no restriction}} = \left(1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}\right)\left(1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}\right)$$

Any combination of two terms will yield a different distinct possibility. Therefore, it is possible to express the answer in terms of

$$Z \equiv \left(1 + 6e^{-\beta} + 12e^{-2\beta} + 8e^{-3\beta}\right)$$

but we will not be doing that here, time to bash away!

(ii) If two particles are identical bosons, they can take on the same energy, except we have to adjust for overcounting. With no restrictions:

- Energy 0: 1 way.
- Energy 1: $1 \cdot 6 = 6$ way.
- Energy 2: $1 \cdot 12 + \binom{6}{2} + 6 = 33$ ways.
- Energy 3: $1 \cdot 8 + 6 \cdot 12 = 80$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2} + 12\right) = 126$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.
- Energy 6: $\binom{8}{2} + 8 = 36$ ways.

With restrictions:

- Energy 2: $\binom{6}{2} + 6 = 21$ ways.
- Energy 3: $6 \cdot 12 = 72$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2} + 12\right) = 126$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.
- Energy 6: $\binom{8}{2} + 8 = 36$ ways.

So:

$$P_2 = 1 - \frac{21e^{-2\beta} + 72e^{-3\beta} + 126e^{-4\beta} + 96e^{-5\beta} + 36e^{-6\beta}}{1 + 6e^{-\beta} + 33e^{-2\beta} + 80e^{-3\beta} + 126e^{-4\beta} + 96e^{-5\beta} + 36e^{-6\beta}}$$

(iii) If two particles are identical fermions, they can take on the same energy (except ground state), except they can't be in the same state.

- Energy 1: $1 \cdot 6 = 6$ ways.
- Energy 2: $1 \cdot 12 + 1 \cdot \binom{6}{2} = 27$ ways.
- Energy 3: $1 \cdot 8 + 6 \cdot 12 = 80$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2}\right) = 114$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.

- Energy 6: $\binom{8}{2} = 28$ ways.

With restrictions:

- Energy 2: $\cdot\binom{6}{2} = 15$ ways.
- Energy 3: $6 \cdot 12 = 72$ ways.
- Energy 4: $6 \cdot 8 + \left(\binom{12}{2}\right) = 114$ ways.
- Energy 5: $12 \cdot 8 = 96$ ways.
- Energy 6: $\binom{8}{2} = 28$ ways.

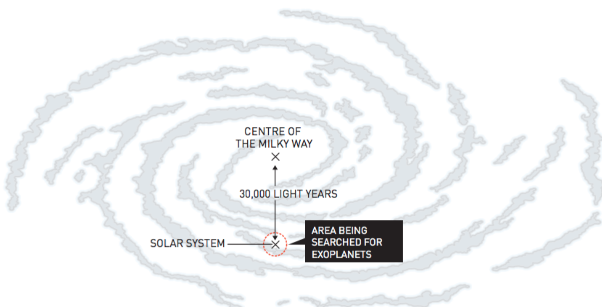
$$P_3 = 1 - \frac{15e^{-2\beta} + 72e^{-3\beta} + 114e^{-4\beta} + 96e^{-5\beta} + 28e^{-6\beta}}{6e^{-\beta} + 27e^{-2\beta} + 80e^{-3\beta} + 114e^{-4\beta} + 96e^{-5\beta} + 28e^{-6\beta}}$$

(d) For high temperature: fermion ¿ distinct ¿ boson. As well as including the math, justification needs to be on the lines of every state having an equal probability, and thus we want the ratio between desired and total states.

For low temperatures, it's reversed. Particles will tend to settle in the lowest energy state (which is what we desire), and distinct particles have more ways to do it than identical bosons. Fermions cannot have two particles in the same state, further decreasing its chances.

Problem 8: Radiation (40 pts)

The Nobel Prize in Physics 2019 was awarded for providing a new understanding of the universe's structure and history, and the first discovery of a planet orbiting a solar-type star outside our solar system.



The Sun is one of several hundred billion stars in our home galaxy, the Milky Way, and there should be planets orbiting most of those stars. So far, astronomers have discovered over 4,000 planets around other stars and they are continuing their search in the area of space closest to us.

Since ancient times, humans have speculated whether there are worlds like our own, with points of views at the extremes expressed thousands of years ago. In modern times, the possibility of observing planets orbiting stars other than the Sun was proposed more than 50 years ago, and has grown into a vast and ever-expanding theory to make the evolution of the universe more clear to us than ever before. In 1995, the very first discovery of a planet outside our solar system, an exoplanet, orbiting a solar-type star was made. This discovery challenged our ideas about these strange worlds and led to a revolution in astronomy. The more than 4,000 known exoplanets are surprising in their richness of forms, as most of these planetary systems look nothing like our own, with the Sun and its planets. These discoveries have led researchers to develop new theories about the physical processes responsible for the birth of planets.

(Taken from the Nobel Prize in Physics 2019 summary, and the Laureates' popular science and scientific views.)

In this problem, we analyse and create a model for a system of two fictitious celestial bodies: an exoplanet and a solar-type star. Unless specified otherwise, consider the two bodies to be solely in each other's gravitational influence and rotate about their barycentre. In the three parts that follow, we will model the physics of a star, of the star-planet model, and the planet respectively.

Part A

The star, with mass $M_s = 2M_\odot$ (twice the mass of our Sun) and radius $R_\odot$ uses nuclear fusion reactions to provide pressure against gravity and electron degeneracy pressure, so as to maintain hydrostatic equilibrium in the star. As long as the hydrostatic equilibrium is preserved, the star is said to be in "main sequence". However, once the energy from the reactions taking place in its core start running out, the star's outer layers swell out to form a red giant. The core of the star (having a radius R_c) starts to shrink, becoming hot and dense; the temperature of the core rises to over a 100 billion degrees, and the pressure from the proton-proton interactions in the core exceeds that of gravity, causing the core to recoil out from the heart of the star in an explosive shock wave. In one of the most spectacular events in the Universe, the shock propels the material away from the star in a tremendous explosion called a supernova. The material spews off into interstellar space.

Being solar-type, this star has the same proton-proton nuclear fusion chain reaction as our Sun: essentially, this is conversion of four protons (mass of a proton is m_P) into 1 He nucleus having mass m_{He}. The star is said to have a "stable lifetime" as long as it is in its "main sequence". The energy emitted by the star passing a sphere of radius r per unit time is $P(r)$, constant over time and the surface of the imaginary sphere of radius r. The density of the exoplanet having radius r_{E} is a constant, ρ, and it orbits around the star in a circular orbit of radius r_{SE}. Neglect any convection effects in the star.

(a) **(4 pts)** Treating the solar-type star as a perfect black-body, estimate the temperature of the surface of the star $T_\odot$ (assumed in thermal equilibrium) by integrating over all frequencies using Planck's distribution for the energy density (defined as the energy per unit volume for a given frequency interval (ν, $\nu + \mathrm{d}\nu$):

$$u(\nu)\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{\mathrm{d}\nu}{(e^{\frac{h\nu}{kT_\odot}} - 1)}$$

where the constants h and k have their usual meanings. For this part only, note that the energy flux from the star onto the exoplanet is J_0. You can use $T_\odot$ as the surface temperature in the later parts.

(b) **(8 pts)** Estimate an expression for, during the main sequence of the star:

(i) the number of protons being fused together per second.

(ii) the stable lifetime of the star, assuming $\eta = 1\%$ mass of the star can undergo nuclear fusion. The change in the temperature or size of the star is insignificant. Assume the only fusion is between protons.

(c) **(8 pts)** Find the temperature gradient $\mathrm{d}T(r)/\mathrm{d}r$ of the star as a function of the radial distance r from the center, such that $R_c \leq r \leq R_\odot$ if the star is in its main sequence, or in a hydrostatic equilibrium. Neglect any quantum-mechanical pressure effects such as electron degeneracy pressure, and assume that pressure from electromagnetic radiation is much larger than any other pressure. State all assumptions.

(d) **(1 pts)** What is the temperature at $r = R_c$, the outermost layer of the core?

(e) **(2 pt)** From experiments, it was found that the temperature gradient of the star is actually

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{9kGM_\odot cP(r)}{128\pi^2\sigma r^2 T^3}$$

Here the modulus of k is one, and has appropriate dimensions. For what value of $P(R_\odot)$ (P_r evaluated at the surface of the star) will the star's main sequence end, leading to the formation of a supernova?

Part B

In this part, we will analyse the radiation effects from the star onto the exoplanet. Assume only black body radiation from the star on the exoplanet. No light is absorbed in the region between the star's and the exoplanet's surface.

(f) **(5 pts)** The distance between the star and the exoplanet is r_{SE}. For this part, assume the surface of the exoplanet has a constant and uniform reflectance γ. What is the force exerted by the radiation from the star on the exoplanet? For the exoplanet's gravitational force to completely balance out the radiation force, how large must the radius of the exoplanet r_E be? Comment on your results and their feasibility.

Part C

(g) **(2 pt)** Find the temperature T_E of the outermost surface of the planet, assumed constant over the whole surface from (a). Assume the planet's surface to be a perfect black body.

(h) **(10 pts)** Model the exoplanet to be made up of N concentric shells equally spaced across the volume of the planet. Between the shells is a peculiar kind of thick type of tectonic rocks which allow no emission, reflection or absorption of energy. However, absorption or emission of radiation energy may take place. The emissivity of all the shells are the same, and are equal to ε, constant and uniform over a surface. Reflection, emission and absorption of any energy due to radiation from the shells, however, may take place. Assume all conduction and convection effects to also be negligible. The temperature of the exoplanet as a function of r is represented by

$$T(r) = T_0\left(1 - \frac{n}{10N}\right)$$

where n is the n^{th} shell from the centre of the planet and T_0 is an appropriate constant as calculated from the previous part (which is unknown, meaning that you need to answer in any variables calculated before). Calculate the total thermal energy due to radiation falling on the outermost shell per unit time. The planet is maintained in a state of thermal equilibrium; this is done by an atmospheric material that allows a fraction (β, which is unknown) of energy from the star falling on the exoplanet. This material only absorbs a fraction of energy it receives from the star. Do NOT assume any such effects for any of the other parts, since they are meant to be crude estimates of the actual calculation. Also compute β.

Solution 8:

(a) Using Planck's distribution,

$$u(\nu) = \int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{\mathrm{d}\nu}{(e^{\frac{h\nu}{kT_\odot}} - 1)}$$

This integration is well known and equal to $\zeta(4) = \pi^4/15$.

Returning to our problem,

$$u(\nu) = \frac{8\pi^5 k {T_\odot}^4}{15h^3c^3}$$

The energy flux onto the exoplanet is

$$J_0 = cu\left(\frac{\Omega}{4\pi}\right)$$

where $\Omega = \frac{\pi {R_\odot}^2}{r_{\text{SE}}^2}$. So

$$J_0 = c \times \frac{8\pi^5 k {T_\odot}^4}{15h^3c^3} \times \frac{\pi {R_\odot}^2}{4\pi r_{\text{SE}}^2} \Rightarrow \boxed{T_\odot = \left(\frac{60{J_0}^3 c^2 {r_{\text{SE}}}^2}{8\pi^5 k {R_\odot}^2}\right)^{\frac{1}{4}}}$$

We get the same result by applying Stefan-Boltzmann's radiation law for a black body, since this is how the law is derived.

(b) (i) In a single nuclear fusion reaction, four protons fuse into one helium nucleus. Let $\mathcal{N}$ be the rate of such fusion reactions (per second), and $P(R_\odot) = 4\pi\sigma {R_\odot}^2 {T_\odot}^4$ be the rate of total energy emitted by the star at its surface. By Einstein's law of mass energy conservation, we have

$$P(R_\odot) = \mathcal{N}(4m_P - m_{\text{He}})c^2 = 4\pi\sigma {R_\odot}^2 {T_\odot}^4$$

which gives

$$\boxed{\mathcal{N} = \frac{4\pi\sigma {R_\odot}^2 {T_\odot}^4}{(4m_P - m_{\text{He}})c^2}}$$

(ii) Let the stable lifetime of the star be T_0. The total number of fusion reactions in the lifetime is then (roughly) $\mathcal{N} \times T_0$, and we have the following expression by substituting $\mathcal{N}$ from (i):

$$\mathcal{N} \times T_0 = \frac{2M_\odot \eta}{4m_p} \Rightarrow \boxed{T_0 = \frac{2M_\odot \eta c^2}{4\pi\sigma {R_\odot}^2 {T_\odot}^4}\left[1 - \frac{m_{\text{He}}}{4m_p}\right]}$$

(c) Consider a circular strip between radial distances r and $r + \mathrm{d}r$ from the centre of the star. For hydrostatic equilibrium to be established, the net force on this strip must be zero, or the pressure forces and gravitational forces on the layer must exactly balance. Let $m(r)$ and $\rho(r)$ be the mass and density of a sphere of radius r respectively. A pressure force of $P(r)$ on this sphere. We have from force balance:

$$-\frac{Gm(r)\rho(r)4\pi r^2 \mathrm{d}r}{r^2} = [P(r + \mathrm{d}r) - P(r)]4\pi r^2$$

$$\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}$$

Note that radiation pressure can be written as $P_r = \frac{4\sigma T^4}{3c}$ by Stefan Boltzmann's law. Using this with the pressure gradient equation, we can write the temperature gradient of the star as:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{3c}{16\sigma T^3}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{3Gm(r)\rho(r)c}{16\sigma T^3 r^2} = \boxed{-\frac{9G{M_\odot}^2 rc}{16\pi\sigma {R_\odot}^6 T^3}}$$

(d) To find the temperature at any point on the star, we solve the gradient equation obtained in (c). Applying the boundary condition $T(R_\odot) = T_\odot$ and separating variables, we have

$$T(r) = \left[T_\odot^4 + \frac{9GM_\odot^2 c}{16\pi\sigma R_\odot^6}(R_\odot^2 - r^2)\right]^{\frac{1}{4}}$$

At $r = R_c$, we have

$$\boxed{T(R_c) = \left[T_\odot^4 + \frac{9GM_\odot^2 c}{16\pi\sigma R_\odot^6}(R_\odot^2 - R_c^2)\right]^{\frac{1}{4}}}$$

(e) From the problem statement, we know that the experiments suggest the following:

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{9kGM_\odot cP(r)}{128\pi^2\sigma r^2 T^3}$$

The main sequence of the star lasts as long as pressure from the gravitational force is greater than the net radiation pressure.

Comment (not for grading purposes):

Throughout the problem, density of the star is assumed constant and equal to $\rho(r) = \frac{6M_\odot}{4\pi R_\odot{}^3}$.

Part (a): The integration was required to be carried out in the examination. A well known method to do this is to split the integral using by parts and express it as the limit of a sum.

Part (c): It is assumed that the pressure balancing the gravitational pressure arises from radiation. There is pressure due to the gas/fuel in the star and a few quantum mechanical effects, but they can be neglected for the sake of this problem.

Problem 9: Piston Gun (40 pts)

In this problem, we examine a model for a certain type of gun that works by using the expansion of a gas to propel a bullet. We can model the bullet as a piston. Since we are assuming atmospheric pressure is negligible, we can assume that the whole setup is in a vacuum. Also, the gun is insulated.

An ideal monatomic gas of initial temperature T_0 is inside a long cylindrical container of cross-section area A. One side of the container is a wall, while the other side is a piston of mass M that can slide freely along the container without friction. The total mass of the gas is m, and it is made up of N particles. Initially, the piston is at rest and a distance L_0 away from the opposite wall. Then, the piston is released. After a time t, the piston moves at a speed v. Assume that throughout the process, the particles on average move very fast.

(a) **(5 pts)** Assume that m is negligible. Find v.

(b) **(6 pts)** From now on, do not assume that m is negligible.

Find the time at which the pressure at the wall opposite the piston changes. Also, does it increase or decrease? State all assumptions.

(c) **(14 pts)** From now on, assume t is much smaller than the mean free time of the particles of the gas, and L_0 is much smaller than the mean free path. (During this time interval t, assume that all the particles still collide many, many times with the walls, but they don't collide with each other.) Find v.

(d) **(6 pts)** Find the recoil impulse of the gun over the time t.

(e) **(9 pts)** Let $r > 1$ be a dimensionless parameter. Suppose at time t, the piston is a distance rL_0 away from the wall; then the piston is stopped, and the gas is allowed to come to equilibrium (after a time much greater than the mean free time). Find the total entropy change (throughout the whole process) of the gas in terms of r, and verify the Second Law of Thermodynamics.

Solution 9:

(a) Since $m \ll M$, the expansion of the gas can be treated as a reversible process, because pressure, density, etc. will be uniformly distributed. The process is also adiabatic because the container is insulated. If x is the position of the piston, we have px^γ is constant, where p is the pressure and $\gamma = \frac{C_p}{C_v} = \frac{5}{3}$ for monatomic gas. Also, Newton's 2nd Law gives $pA = M\ddot{x}$. We have $\ddot{x} = \frac{p_0 A L_0^\gamma}{M x^\gamma}$, where $p_0 = \frac{NkT_0}{AL_0}$. To find v, we need to solve this differential equation and plug in the boundary conditions of $x(0) = L_0$ and $\dot{x}(0) = 0$. Since the final answer is very complicated, we have decided to award (almost) full credit for those who end up with correct differential equation and boundary conditions (and have accurate, thorough reasoning).

(b) Now, we cannot neglect m anymore, so the pressure will not be uniform throughout the gas. In fact, a pressure wave will start at the piston and travel to the opposite wall at speed $c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{5NkT_0}{3m}}$.

The time it takes is $\frac{L_0}{c} = \boxed{L_0\sqrt{\frac{3m}{5NkT_0}}}$.

(c) Let $L(t)$ be the position of the piston as a function of time, where $L(0) = L_0$. Let $v = \dot{L}$. Let the x direction be along the cylinder's axis. Consider a particle in the gas with speed $v_x(t)$ in the x direction. We will show that Lv_x is a conserved quantity. Since the mean free time is large, we can assume the particles rarely collide with each other. Since the gas is monatomic, the particle makes elastic collisions with the walls. When it collides with a stationary wall, the particle's speed stays the same. However, if it moves at speed v_x and collides with the piston moving at speed v, it rebounds with a new speed $v_x - 2v$, which can be seen by shifting to the piston's frame and noting that the particle's mass is much smaller than the piston's. Thus, its decrease in speed is $2v$. Since it is moving very fast, on average, it loses speed $2v$ every time it takes to go back and forth, which is $\frac{2L}{v_x}$. We can write $\frac{dv_x}{dt} = -\frac{2v}{\frac{2L}{v_x}}$. This simplifies to $L\dot{v_x} + vv_x = 0$. Integrating, we get $Lv_x = c$ where c is an integration constant. Plugging in initial conditions, we have $Lv_x = L_0 v_{x0}$, where v_{x0} is the initial x-velocity of the particle.

Next, we write the conservation of energy equation after time t. For simplicity, we redefine m as the mass of each particle.

$$\int_{-\infty}^{\infty} \frac{1}{2} Nm(v_{x0}^2 - v_x^2) f(v_{x0})\,\mathrm{d}v_{x0} = \frac{1}{2} Mv^2$$

where $f(v_{x0}) = \sqrt{\frac{m}{2\pi kT_0}} e^{-\frac{mv_{x0}^2}{2kT_0}}$ is the Maxwell-Boltzmann distribution in the x direction. Using the fact that $Lv_x = L_0 v_{x0}$ and substituting for f, we get

$$\int_{-\infty}^{\infty} Nmv_{x0}^2 (1 - \frac{L_0^2}{L^2}) \sqrt{\frac{m}{2\pi kT_0}} e^{-\frac{mv_{x0}^2}{2kT_0}}\,\mathrm{d}v_{x0} = Mv^2$$

Integrating and simplifying, we get

$$\dot{L} = \sqrt{\frac{NkT_0}{M}} \sqrt{1 - \frac{L_0^2}{L^2}}$$

$$\frac{L}{\sqrt{L^2 - L_0^2}}\mathrm{d}L = \sqrt{\frac{NkT_0}{M}}\mathrm{d}t$$

Integrating and plugging in initial conditions gives

$$\sqrt{L^2 - L_0^2} = \sqrt{\frac{NkT_0}{M}} t$$

$$L = \sqrt{L_0^2 + \frac{NkT_0}{M} t^2}$$

Finally, we have $\boxed{v = \frac{NkT_0 t}{\sqrt{M^2 L_0^2 + NkT_0 M t^2}}}$

(d) The recoil impulse is simply the momentum of the gas + piston. The center of mass of the gas moves at half the speed of the piston, so we have the total momentum $Mv + \frac{mv}{2} = \boxed{\left(M + \frac{m}{2}\right) \frac{NkT_0 t}{\sqrt{M^2 L_0^2 + NkT_0 M t^2}}}$

(e) After a long time, the pressure will again become uniform, and we can use the formula for entropy of an ideal gas: $Nk \ln PV_0^\gamma$. To do this, we need to first realize that when the piston is stopped, $\frac{1}{2}Mv^2$ of energy is lost, and combining this fact with the ideal gas law gives us enough information to solve for the final pressure of the gas. After calculation, we get $\Delta S = \boxed{Nk\left(\frac{3}{2}\ln\left(\frac{2}{3} + \frac{1}{3r^2}\right) + \ln r\right)}$. Note that there are no surroundings (vacuum). Also, it is easy to see ΔS is positive if one finds that the derivative with respect to r is positive for $r > 1$. Thus, the Second Law of Thermodynamics is verified.

Problem 10: Magnetostatics (62 pts)

Part A

In 3-D space, a permeable medium covers the region $x > 0$, while the rest of the space is vacuum. The medium's relative magnetic permeability is $\mu_r > 1$. A magnetic dipole with dipole moment m is placed a distance d away from the permeable medium, at position $(-d, 0, 0)$. The dipole is pointed towards the +x direction. Treat the dipole as ideal (point-sized).

(a) **(10 pts)** Find the force required to keep the dipole in place.

(b) **(3 pts)** How much work does it take to slowly pull the dipole from its original position to infinity (at $x = -\infty$)?

(c) **(5 pts)** How much work does it take to slowly rotate the dipole from its original orientation to one that makes an angle θ with the +x-axis?

After the dipole is rotated an angle θ, a superconducting ring with radius R and self-inductance L is brought in from infinity (with initially no current). It is placed so that the dipole is located at its center and its axis is the x-axis. Assume that $R \gg d$.

(d) **(16 pts)** Find the current I in the ring.

(e) **(8 pts)** Find the force required to hold the dipole in place (not the torque).

Part B

From now on, there is no permeable medium. Ignore any radiation loss for all parts.

(f) **(7 pts)** The dipole (mass M), starts at a distance h from the centre of the ring (kept fixed) and pointed towards the centre of the ring (along its axis), and is projected with a small velocity v_0 towards the centre. Find its speed v as a function of h. Ignore gravity.

(g) **(7 pts)** Consider another scenario, in which the dipole is placed on the axis of a thin infinite magnetic tube with surface conductivity σ (defined as the ratio of surface current density and the electric field) and radius R, placed at an arbitrary location inside it. (You may neglect the self inductance of the solenoid for the sake of this part.) We find that the motion of the dipole in this case is damped. Find the damping parameter of this motion. (Damping parameter is defined as the ratio of the resistive force to the speed.) Ignore gravity.

(h) **(6 pts)** Determine the terminal velocity of the magnet, assuming that it now falls under gravity. The tube may be considered infinitely long for all calculation purposes in this part.

Solution 10: For permeable media, two of Maxwell's equations become

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}}$$

where $\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{J}_{free} = 0$ because there is no free current. Note that $\mu = \mu_r\mu_0$. Thus, the equations become $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{H} = 0$. Applying these to the boundary, we get the following boundary conditions:

$$B_{1,\perp} = B_{2,\perp} \tag{1}$$
$$H_{1,\parallel} = H_{2,\parallel} \tag{2}$$

where region 1 is $x < 0$ and region 2 is $x > 0$. We claim that if a magnetic monopole of magnetic charge q is placed a distance d away from the permeable medium instead of a dipole,
(1) in the region $x < 0$, the magnetic field is equivalent to that of the original monopole and a magnetic monopole placed at $(d, 0, 0)$ with magnetic charge $q_1 = -\frac{\mu_r - 1}{\mu_r + 1} q_0$ (2) in the region $x > 0$, the magnetic field is equivalent to that of the original monopole at its current position but with charge $q_2 = \frac{2\mu_r}{\mu_r + 1} q_0$.
To show this, note that as long as we show that the boundary conditions are satisfied, we are done by the uniqueness theorem because the field in the rest of the space satisfies Laplace's equation. For the region

$x < 0$, the boundary at the real magnetic monopole is already satisfied. Now, the only boundary we need to consider is the interface $x = 0$. Consider a point P on the plane $x = 0$ such that the line from $(-d, 0, 0)$ to P forms an angle θ with the x-axis. Define B_i to be the magnitude of the field at P from q_i. According to the claim, the magnetic field near P in region 1 has a magnetic field component perpendicular to the interface of $B_{1,\perp} = B_0 \cos\theta - B_1 \cos\theta$. This must be equal to the perpendicular component in region 2, so we have

$$B_0 \cos\theta - B_1 \cos\theta = B_2 \cos\theta \tag{3}$$

Now, the parallel component of $\mathbf{H}$ in region 1 is $\frac{B_0}{\mu_0}\sin\theta + \frac{B_1}{\mu_0}\sin\theta$, which must be equal to the parallel component of $\mathbf{H}$ in region 2, so

$$\frac{B_0}{\mu_0}\sin\theta + \frac{B_1}{\mu_0}\sin\theta = \frac{B_2}{\mu}\sin\theta \tag{4}$$

Solving the two equations gives

$$B_1 = -\frac{\mu_r - 1}{\mu_r + 1}B_0 \tag{5}$$

$$B_2 = \frac{2\mu_r}{\mu_r + 1}B_0 \tag{6}$$

Noting that $B_i \propto q_i$, we see that the claim is true.
We apply the claim to the problem via superposition. We treat the magnetic dipole as two magnetic monopoles of magnetic charges q and $-q$ that are a small distance s away from each other. Note that $m = qs$.

(a) If the dipole is at $(-d, 0, 0)$ and pointed towards the +x direction, the field in region 1 is equivalent to that of the same dipole superposed with another dipole at $(d, 0, 0)$ pointing in the +x direction with moment $m_1 = q_1 s = \frac{\mu_r - 1}{\mu_r + 1}m$. The field produced by the latter magnetic dipole at position of the original dipole is given by $\frac{\mu_0 m_1}{2\pi r^3}$, where $r = 2d$. The magnetic force on the original dipole is given by

$$F = m\frac{\partial B_x}{\partial x} = \frac{3\mu_0 m m_1}{2\pi r^4} = \frac{3\mu_0 m m_1}{32\pi d^4} = \boxed{\frac{3\mu_0(\mu_r - 1)m^2}{32\pi(\mu_r + 1)d^4}} \tag{7}$$

Note that the force is attractive.

(b) We integrate the force from (a) to get the work done:

$$W = \int_d^\infty \frac{3\mu_0(\mu_r - 1)m^2}{32\pi(\mu_r + 1)x^4}\,\mathrm{d}x = \boxed{\frac{\mu_0(\mu_r - 1)m^2}{32\pi(\mu_r + 1)d^3}} \tag{8}$$

(c) Instead of rotating the dipole from the original orientation, we first bring the dipole to infinity, rotate it, and then bring it back to $(-d, 0, 0)$. From part (b), we have the work it takes to bring the dipole to infinity. Rotating it at infinity requires no work. Finally, to calculate the amount of work required to bring it back, we find the potential energy of the final system. From our claim, we can apply superposition to see that the setup is equivalent to one in which the permeable medium acts like a magnetic dipole at $(d, 0, 0)$ with moment $m_1 = \frac{\mu_r - 1}{\mu_r + 1}$ and pointed in a direction that makes an angle $-\theta$ with the +x-axis. The general formula for dipole-dipole interaction is given by:

$$U = \frac{\mu_0}{4\pi r^3}(\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})) \tag{9}$$

which equals

$$U = \frac{\mu_0}{4\pi(2d)^3}(m m_1 \cos 2\theta - 3 m m_1 \cos^2\theta) = -\frac{\mu_0 m m_1}{32\pi d^3}(\cos^2\theta + 1) \tag{10}$$

However, this is not the work we do to bring it back from infinity; rather, it is twice our work. To see why, imagine an external agent bringing in the imaginary dipole from $+\infty$ as we are bringing in the real dipole from $-\infty$. By Newton's third law, the force they apply is equal and opposite to ours, and so they do the same amount of work as we do. Thus, the work it takes to bring in the dipole from $-\infty$ is

$$W = -\frac{\mu_0 m m_1}{64\pi d^3}(\cos^2\theta + 1) \tag{11}$$

In total, the amount of work done is

$$W = \boxed{\frac{\mu_0(\mu_r - 1)m^2\sin^2\theta}{64\pi(\mu_r+1)d^3}} \tag{12}$$

(d) By similar reasoning from the previous parts, we can see that the image of the ring with current I will be a ring centered at $x = d$ with current $\frac{\mu_r-1}{\mu_r+1}I$ in the same direction.

Claim 1: The mutual inductance between the real ring and the image dipole is given by $\frac{\mu_0 R^2\cos\theta}{2(R^2+4d^2)^{\frac{3}{2}}} * \frac{\mu_r-1}{\mu_r+1}A$, where A is the area of the dipole.

Proof: By the mutual inductance reciprocity theorem ($M_{12} = M_{21}$), we just need to calculate the flux through the dipole over a current I in the ring. Since the dipole is ideal, we can find the field at the dipole and do $\Phi = BA\cos\theta$. After plugging in well-known expression $B = \frac{\mu_0 R^2}{2(R^2+x^2)^{\frac{3}{2}}}$, where $x = 2d$, we get the desired answer.

A natural corollary of this claim is that the mutual inductance between the real ring and the real dipole is $\frac{\mu_0\cos\theta}{2R}m$ **Claim 2**: The mutual inductance between the real ring and the image ring is given by $\mu_0 R\ln\frac{R}{d}$. **Proof**: Consider the field lines from the ring. Since $d << R$, the field lines near the edge of the ring are circular, so the flux through the image ring from the real ring is the same as the flux through a circle coplanar with the real ring with radius $R - d$. This is because of the fact that magnetic field has zero divergence. To find the flux through the circle of radius $R - d$, we use the fact

$$\iint_S \mathbf{B}\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l}$$

where $\mathbf{A}$ is the vector potential. By rotational symmetry, we only need to calculate the tangential component of $\mathbf{A}$ at a point $R - d$ from the center, and then multiply by $2\pi(R - d)$. We have

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{Id\mathbf{l}}{r} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{R\cos\theta d\theta}{\sqrt{R^2+(R-d)^2-2R(R-d)\cos\theta}}\hat{\theta}$$

We simplify to get

$$A_\theta = \frac{\mu_0 IR}{2\pi\sqrt{R(R-d)}}\int_0^{\pi}\frac{\cos\theta}{\sqrt{2+\frac{d^2}{R^2}-2\cos\theta}}d\theta \approx \frac{\mu_0 I}{2\pi}\int_0^{\pi}\frac{\cos\theta}{\sqrt{2+\frac{d^2}{R^2}-2\cos\theta}}d\theta$$

Now, we realize that the integral diverges if $d = 0$. Thus, since $d << R$, we see that the integral is dominated by the region where θ is close to zero. We can use the small angle approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$. We simplify to get

$$A \approx \frac{\mu_0 I}{2\pi}\int_0^{\pi}\frac{1}{\sqrt{\theta^2+\frac{d^2}{R^2}}}d\theta = \frac{\mu_0 I}{2\pi}\ln(\sqrt{(\theta * \frac{R}{d})^2+1}+\theta * \frac{R}{d})|_0^{\pi} \approx \frac{\mu_0 I}{2\pi}\ln\frac{R}{d}$$

Finally, we multiply by $2\pi R$ and divide by I to get the desired mutual inductance. Note: the result can also be achieved using Grover's formula.

The flux through the superconducting ring must remain zero. The flux contributed by the ring itself is simply LI. The flux contributed by each dipole is the mutual inductance over the area times the dipole moment. The flux contributed by the image ring is the mutual inductance times the current in the image ring. Combining all the terms and setting it to zero gives: $LI - \frac{\mu_0 \cos\theta}{2R}m - \frac{\mu_0 R^2 \cos\theta}{2(R^2+4d^2)^{\frac{3}{2}}} * \frac{\mu_r - 1}{\mu_r+1}m + \mu_0 R \ln\frac{R}{d} * \frac{\mu_r-1}{\mu_r+1}I = 0$ Solving, we get $\boxed{I = \frac{\mu_0 m \cos\theta}{2} \cdot \frac{\frac{1}{R} - \frac{\mu_r-1}{\mu_r+1} * \frac{R^2}{(R^2+4d^2)^{\frac{3}{2}}}}{L + \mu_0 R \frac{\mu_r-1}{\mu_r+1}\ln\frac{R}{d}}}$

(e) The force on a dipole is given by $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{B})$. We can break this force up into components. The x-component is given by $F_x = p_x \frac{\partial B_x}{\partial x} + p_y \frac{\partial B_x}{\partial y}$, where the dipole is in the x-y plane. Note that the ring contributes no gradient of $\mathbf{B}$ field at the dipole, so we only need to consider the field from the image ring and image dipole. First we find the force on the dipole due to the image ring. The field from the image ring is given by $B = \frac{\mu_0 I_1 R^2}{2(R^2+(2d)^2)^{\frac{3}{2}}}$, so the gradient in the x direction is $\frac{3\mu_0 I_1 R^2 d}{(R^2+(2d)^2)^{\frac{5}{2}}}$. The x-component of the field from the image ring doesn't change significantly in the y-direction, by symmetry, so we get $\frac{\partial B_x}{\partial y} = 0$.

Now we consider the y-component $F_y = p_x \frac{\partial B_y}{\partial x} + p_y \frac{\partial B_y}{\partial y}$. The y-component of the field from the image ring along its axis is 0, so $\frac{\partial B_y}{\partial x} = 0$. To find $\frac{\partial B_y}{\partial y}$, we use the fact that $\nabla\cdot\mathbf{B} = 0$, so we get $\frac{\partial B_x}{\partial x} + 2\frac{\partial B_y}{\partial y} = 0$, where here we are only considering the field from the image ring. Thus, $\frac{\partial B_y}{\partial y} = -\frac{\partial B_x}{2\partial x} = -\frac{3\mu_0 I_1 R^2 d}{2(R^2+(2d)^2)^{\frac{5}{2}}}$.

It remains to find the force between the image dipole and the real dipole. The dipole-dipole interaction is given by equation (9). To find the x-component of the force F_x, we can calculate the derivative of the potential energy in the x direction. Moving the dipole in the x direction only changes r, so we have $F_x = \frac{dU}{dr} = -\frac{3\mu_0}{4\pi r^4}(mm_1\cos(2\theta) - 3mm_1\cos^2\theta) = \frac{3\mu_0}{4\pi(2d)^4}mm_1(\cos^2\theta+1) = \frac{3\mu_0}{64\pi d^4}mm_1(\cos^2\theta+1)$.

To find the y-component of the force F_y, we can calculate the derivative of the potential energy in the y direction. Moving the dipole in the y direction does not change r to first order, but does change $\hat{\mathbf{r}}$. Thus, $F_y = -\frac{dU}{dy} = -\frac{3\mu_0}{4\pi r^4}(2mm_1\cos\theta\sin\theta) = -\frac{3\mu_0}{32\pi d^4}mm_1\cos\theta\sin\theta$.

Finally, we add up all the forces to get

$$\boxed{\mathbf{F} = \mu_0\frac{\mu_r-1}{\mu_r+1}m((-\frac{3IR^2 d\cos\theta}{(R^2+4d^2)^{\frac{5}{2}}} + \frac{3}{64\pi d^4}m(\cos^2\theta+1))\hat{\mathbf{x}} + (\frac{3IR^2 d\sin\theta}{2(R^2+4d^2)^{\frac{5}{2}}} - \frac{3}{32\pi d^4}m\cos\theta\sin\theta)\hat{\mathbf{y}})}$$

, where I is given in part (d).